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A Comparison of DHW Algorithm for Temperature Distribution Calculation with Fourier's Algorithm for Transmission of Heat between Discrete Bodies

Jozef Gembarovic[1,2] **and Jozef Gembarovic, Jr.**[3]

In this article, our damped-heat wave (DHW) algorithm for the calculation of temperature distribution in a homogeneous finite medium is compared with Fourier's algorithm for transmission of heat between discrete bodies.

KEY WORDS: heat conduction; numerical algorithm; temperature distribution.

1. INTRODUCTION

Joseph Fourier's work [1] in formulating the heat conduction in terms of a partial differential equation and developing the methods for solving the equation is well known and recognized as one of the biggest scientific achievements of mankind. Less known is his first attempt to solve the problem of heat transmission between discrete bodies.

Fourier started work on heat conduction sometime between 1802 and 1804 [2]. Inspired by the Laplacian philosophy of action at a distance, he followed an 18th-century technique of developing a discrete model of the continuous phenomenon and initially formulated the heat conduction as an n-body problem. He studied and found the solutions only to the two examples—straight line and circular arrangement of N discrete bodies—and then he stopped. Although he never even mentioned that there was a difficulty, he probably reached the point from which only the investigation of special cases was possible [3]. Fourier abandoned the n-body approach

[1] TPRL, Inc., 3080 Kent Avenue, West Lafayette, Indiana 47906, U.S.A.
[2] To whom correspondence should be addressed. E-mail: gembar@tprl.com
[3] College of Technology, Purdue University, West Lafayette, Indiana 47906, U.S.A.

around 1804, and probably inspired by Biot's work [4], he started to work on a theory of heat conduction in continuous bodies.

Although his first attempt to describe heat transfer using discrete bodies is regarded as a dead end or a blind alley, Fourier never failed to describe it in detail in his 1805 Draft Paper, through his 1807 Essay [5], to the Prize Paper of 1811, and the book *Théorie Analytique de la Chaleur* [6]. As a result of Fourier's lively historical sense, the "Communication of Heat Between Discrete Bodies" is the second largest section of Fourier's book [7] as a monument to his earliest research in the heat conduction problem [8].

Recently, we have described a very simple algorithm for calculation of the temperature distribution in finite one-dimensional bodies [9, 10]. We named it the damped-heat-wave (DHW) algorithm, and it is similar (but not the same) to the above-mentioned Fourier's algorithm for transmission of heat between discrete bodies. In Sect. 2 of this article, we will describe both the Fourier's and the DHW algorithms. In case of the earlier, we tried to follow an original description found in Fourier's book [7], adding only our remarks and formulae for a length representation of the temperatures. In Section 3, the two algorithms will be compared using a simple (but practical) boundary value problem.

2. DESCRIPTION OF ALGORITHMS

2.1. Fourier's Algorithm

Fourier first considered [11] two rectangular bodies of equal mass m, cross-section area A, and thickness Δl of the same material with the same specific heat c, density ρ, and perfect thermal conductivity at different temperatures a and b. He imagined the transmission of heat between the bodies by means of an ideal shuttle mechanism consisting of an infinitesimally small section of thickness δ and mass ω which moves to and fro in a fixed time Δt between the two masses.

If these two bodies are placed in contact, the temperature in each would suddenly become equal to the mean temperature $\frac{1}{2}(a+b)$. Two masses (see Fig. 1) are separated by a very small interval. A thin layer ω of the first is detached so as to be joined to the second, and then it returns to the first immediately after the contact. Continuing thus to be transferred alternately, and at equal small time intervals, the interchanged layer causes the heat of the hotter body to pass gradually into that which is less heated. There are no heat losses from the bodies to ambient. The quantity of heat contained in the thin layer is suddenly added to that of the body with which it is in contact, and a common temperature results

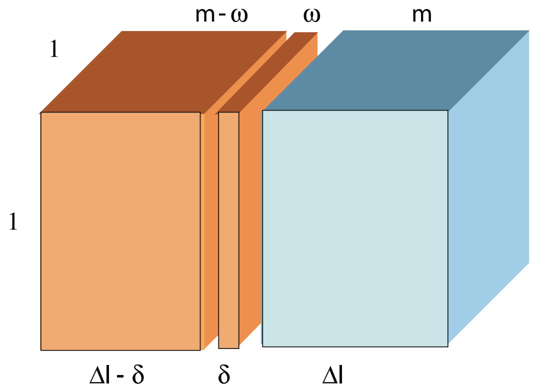

Fig. 1. Heat transfer between two discrete bodies.

which is equal to the quotient of the sum of the quantities of heat divided by the sum of the masses multiplied by the specific heat. Let ω be the mass of the small layer which is separated from the hotter body, whose temperature is a; let θ and ϑ be the variable temperatures which correspond to the time t, and whose initial values are a and b. When the layer ω is separated from the mass m which becomes $m-\omega$, it has the temperature θ, and as soon as it touches the second body with the temperature ϑ, it assumes at the same time with that body a temperature equal to

$$\frac{\vartheta mc+\theta\omega c}{mc+\omega c}=\frac{\vartheta m+\theta\omega}{m+\omega}. \tag{1}$$

The layer ω, retaining the last temperature, returns to the first body, whose mass is $m-\omega$ and temperature is θ. The temperature after the second contact is

$$\frac{\theta(m-\omega)c+((\vartheta m+\theta\omega)/(m+\omega))\omega c}{mc}=\frac{\theta m+\vartheta\omega}{m+\omega}. \tag{2}$$

The variable temperatures θ and ϑ become, after the interval Δt,

$$\theta-(\theta-\vartheta)\frac{\omega}{m} \quad \text{and} \quad \vartheta+(\theta-\vartheta)\frac{\omega}{m}. \tag{3}$$

For the differences we have

$$\Delta\theta=-(\theta-\vartheta)\frac{\omega}{m} \qquad \text{and} \qquad \Delta\vartheta=(\theta-\vartheta)\frac{\omega}{m}. \tag{4}$$

The quantity of heat received in one instant by the second mass is equal to the quantity of the heat lost by the first mass. The quantity of

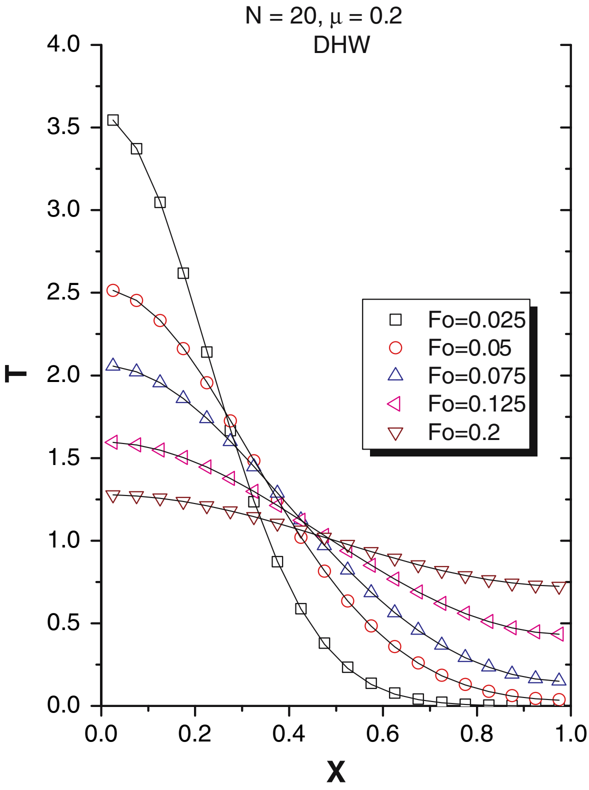

Fig. 2. Temperature distribution in a finite body calculated using the DHW algorithm for $N = 20$, $\mu = 0.2$, and $\Delta Fo = 5 \times 10^{-4}$ at different times $Fo = 0.025$, 0.05, 0.075, 0.15, 2.0. Exact solutions calculated using Eq. (20) are depicted as solid lines.

the heat is, if we assume that all other things being equal, proportional to the actual difference of temperature of the two bodies.

While $m = \Delta l A\rho$ and $\omega = \delta A\rho$, the masses m and ω can be replaced by Δl and δ, respectively. (We will call this substitution a length representation.) Equation (4) is now

$$\Delta\theta = -(\theta - \vartheta)\frac{\delta}{\Delta l} \qquad \text{and} \qquad \Delta\vartheta = (\theta - \vartheta)\frac{\delta}{\Delta l}. \tag{5}$$

The term ω (or δ) represents the velocity of transmission, or the facility with which the heat passes from one of the bodies into the other. Fourier [12] called it the *reciprocal conducibility*. He integrated the temperatures of Eq. (4) and found a formula for the transient temperature of the system.

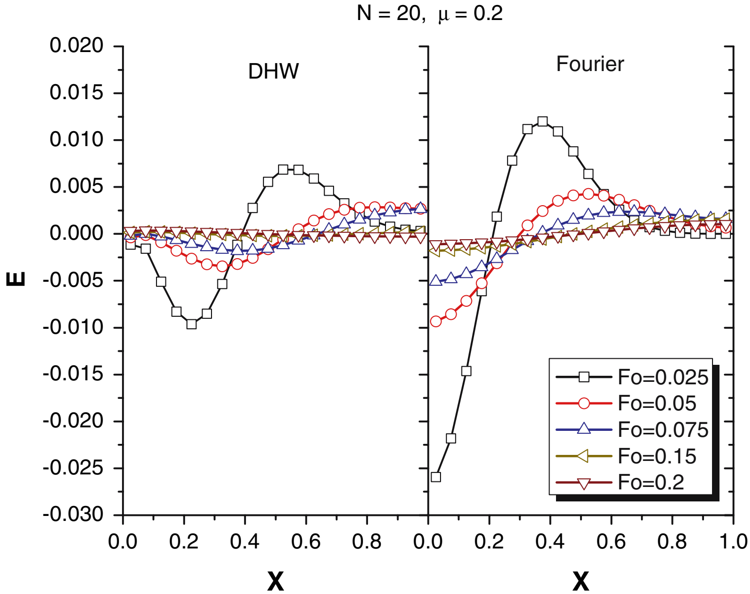

Fig. 3. Error distribution for the DHW and Fourier's algorithm (both for $N=20$, $\mu=0.2$, and $\Delta Fo=5\times10^{-4}$) at different times Fo.

In a general case of N separate equal masses arranged in a straight line, Fourier considered [13] transmission of heat by the same shuttle mechanism as in the case of two bodies only. Infinitesimally thin layers ω move to and fro between successive bodies, all at once, so the situation for inner bodies is not the same as for the two bodies at the boundaries.

Let $\alpha, \beta, \gamma, \ldots, \psi$, be the variable temperatures which correspond to the same time t, and which have succeeded to the initial values $a, b, c, \ldots$. When the layers ω have been separated from the first masses and put in contact with the neighboring masses, the temperatures become

$$\frac{\alpha(m-\omega)}{m-\omega}, \quad \frac{\beta(m-\omega)+\alpha\omega}{m}, \ldots, \quad \frac{m\psi+\chi\omega}{m+\omega}; \tag{6}$$

or

$$\alpha, \quad \beta+(\alpha-\beta)\frac{\omega}{m}, \quad \ldots, \quad \psi+(\chi-\psi)\frac{\omega}{m+\omega}. \tag{7}$$

When the layers ω have returned to their former places, the new temperatures (after the instant dt) are

$$\alpha+(\beta-\alpha)\frac{\omega}{m},\ \beta+(\alpha-2\beta+\gamma)\frac{\omega}{m},\ \ldots,\ \psi+(\chi-\psi)\frac{\omega}{m}, \tag{8}$$

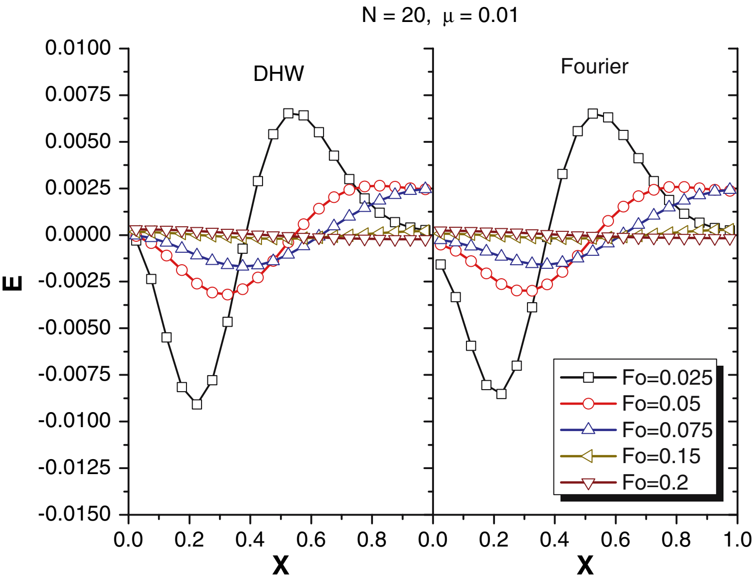

Fig. 4. Error distribution for the DHW and Fourier's algorithm (both for $N = 20$, $\mu = 0.01$, and $\Delta Fo = 2.5 \times 10^{-5}$) at different times $Fo = 0.025$, 0.05, 0.075, 0.15, 2.0.

where the terms with ω^2 are neglected.

If the masses ω and m in Eq. (8) are replaced with the layer's thicknesses δ and Δl, respectively, the temperatures finally become

$$\alpha + (\beta - \alpha)\frac{\delta}{\Delta l},\ \beta + (\alpha - 2\beta + \gamma)\frac{\delta}{\Delta l},\ \ldots,\ \psi + (\chi - \psi)\frac{\delta}{\Delta l}. \tag{9}$$

Finally, Fourier considered N separate equal masses to be placed at equal distances on the circumference of a circle [14]. Heat is transferred by the same shuttle mechanism between the bodies as in the case of N separate equal masses arranged in a line, but the masses are now all considered to be inner. Transient temperatures $\alpha, \beta, \ldots, \psi$ can now be expressed in symmetrical forms,

$$\alpha + (\psi - 2\alpha + \beta)\frac{\omega}{m},\ \beta + (\alpha - 2\beta + \gamma)\frac{\omega}{m},\ \ldots,\\ \psi + (\chi - 2\psi + \alpha)\frac{\omega}{m}, \tag{10}$$

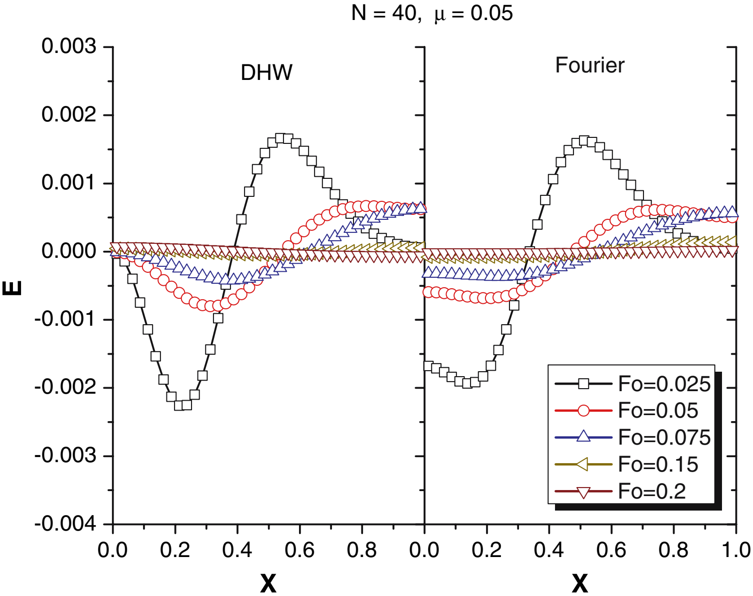

Fig. 5. Error distribution for the DHW and Fourier's algorithm (both for $N=40$, $\mu=0.05$, and $\Delta Fo=3.125\times10^{-5}$) at different times $Fo=0.025$, 0.05, 0.075, 0.15, 2.0.

or in the length representation,

$$\alpha+(\psi-2\alpha+\beta)\frac{\delta}{\Delta l},\ \beta+(\alpha-2\beta+\gamma)\frac{\delta}{\Delta l},\ \ldots,\ \psi+(\chi-2\psi+\alpha)\frac{\delta}{\Delta l}. \tag{11}$$

After an ingenious and original set of manipulations, Fourier found an analytical solution for the temperature of the system and showed that the formula for the discrete bodies arranged in a circle is equal for $N\to\infty$ to the one he found solving the partial differential equation of heat conduction. Then he remarked [15]: "It is not necessary to resort to analysis of partial differential equations in order to obtain the general equation which expresses the movement of heat in a ring. The problem may be solved for a definite number of bodies, and that number may be then supposed infinite. This method has a clearness peculiar to itself, and guided our first researches."

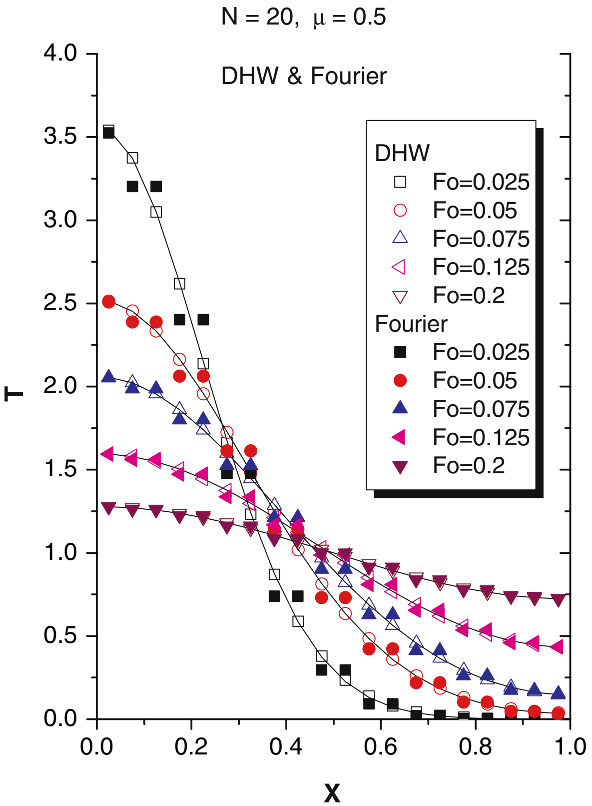

Fig. 6. Temperature distribution in a finite body calculated using the DHW and Fourier's algorithm for $N=20$, $\mu=0.5$, and $\Delta Fo=1.25\times10^{-3}$ at five different times $Fo=0.025$, 0.05, 0.075, 0.15, 2.0. Exact solutions calculated using Eq. (20) are depicted as solid lines.

2.2. DHW Algorithm

In the DHW algorithm for calculation of a temperature distribution [9], a finite homogeneous medium of thickness L is divided into N equal slabs of thickness $\Delta l = L/N$. These slabs are replaced by a perfect conductor of the same heat capacity separated by the thermal resistance $\Delta l/\lambda$, (where λ is the thermal conductivity of the medium), so the temperature within a slab at any given time is constant. Heat propagates through the medium due to a temperature difference between the slabs. A certain portion (given by the inner heat transfer coefficient ξ) of the excessive heat energy moves from one slab to the next one, lowering thus the temperature difference between the two neighbor slabs. This redistribution process

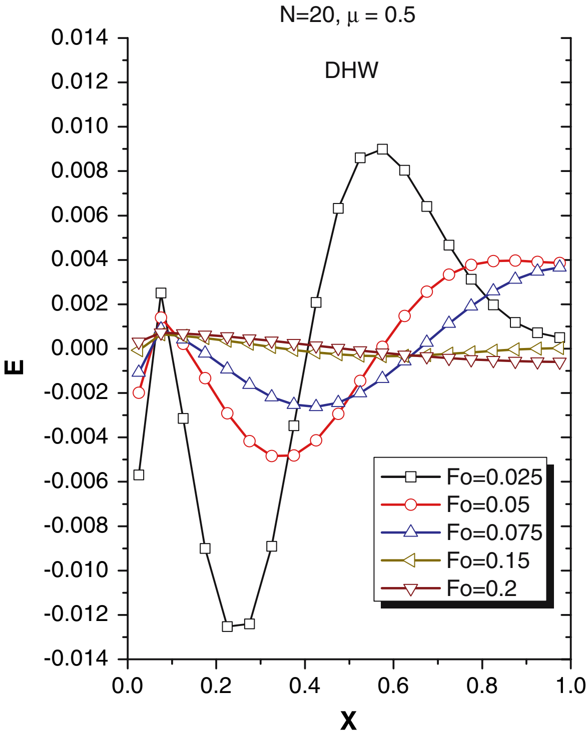

Fig. 7. Error distribution for the DHW algorithm ($N=20$, $\mu=0.5$, and $\Delta Fo=1.25\times10^{-3}$) at different times $Fo=0.025$, 0.05, 0.075, 0.15, 2.0.

(the DHW) starts from the left boundary slab and marches in space from one pair of slabs to another. When the wave reaches the boundary of the medium, it bounces back and moves in the opposite direction in a perpetual manner.

Slab temperatures are $T_{i,m}\equiv T(x_i, t_m)$, where x_i, $(i=0,1,2,\ldots,N-1)$ is a spatial point (middle of the ith slab), and $t_m=m\Delta t$ $(m=0,1,2,\ldots)$ is a discrete time point. The temperature of the boundary slabs is actually changing only after the heat wave finishes one whole loop; therefore, the time step Δt is equal to one loop time interval. The time step Δt is thus divided into $2N$ sub-steps. Despite almost trivial simplicity of this algorithm, the temperature distribution in the medium at time t_{m+1} as a function of the temperature distribution at t_m can be expressed by rather lengthy and complicated formulae [9]:

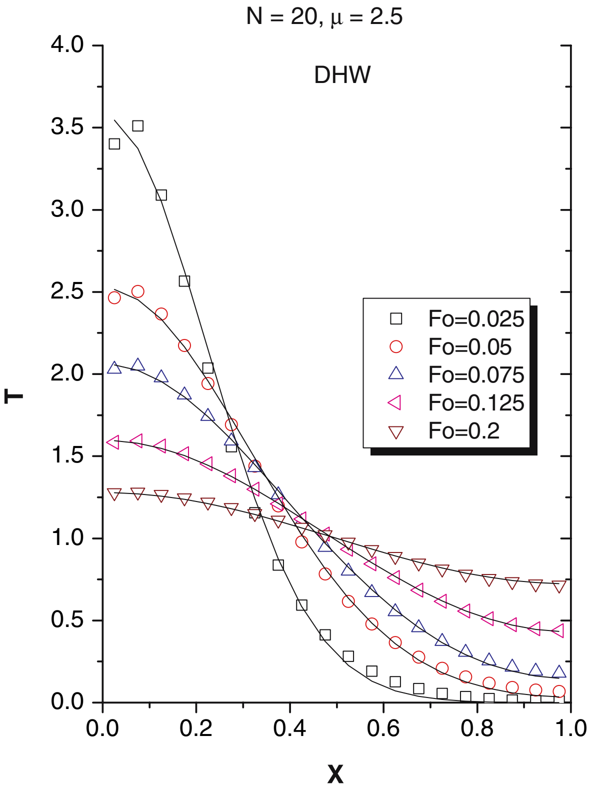

Fig. 8. Temperature distribution in a finite body calculated using the DHW algorithm for $N=20$, $\mu=2.5$, and $\Delta Fo=6.25\times10^{-3}$ at five different times $Fo=0.025$, 0.05, 0.075, 0.15, 2.0. Exact solutions calculated using Eq. (20) are depicted as solid lines.

$$
\begin{aligned}
T_{0,m+1} &= \left[2\frac{1+\xi^{2N-1}}{1+\xi}-1\right]T_{0,m} \\
&\quad +2\frac{1-\xi}{1+\xi}\sum_{j=1}^{N-1}\xi^{j}\left(1+\xi^{2(N-j)-1}\right)T_{j,m}, \\
T_{i,m+1} &= 2\frac{1-\xi}{1+\xi}\left(1+\xi^{2(N-i)-3}\right)\sum_{j=0}^{i-1}\xi^{j}(1-\xi)^{j}T_{j,m} \\
&\quad +\left[(1-\xi)^2\left(2\frac{1+\xi^{2(N-i)-3}}{1+\xi}-1\right)+\xi^2\right]T_{i,m}
\end{aligned}
$$

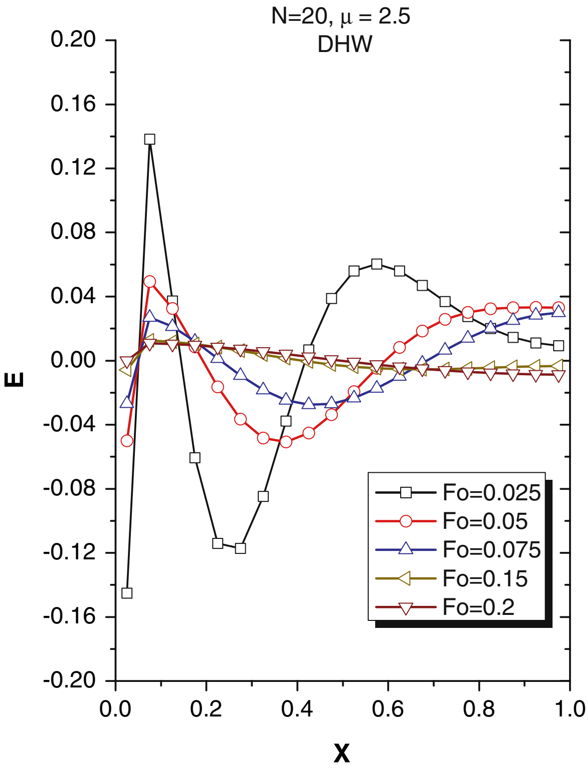

Fig. 9. Error distribution for the DHW algorithm for $N=20$, $\mu=2.5$, and $\Delta Fo=6.25\times10^{-3}$ at five different times $Fo=0.025$, 0.05, 0.075, 0.15, 2.0.

$$+2\frac{(1-\xi)^2}{1+\xi}\sum_{j=0}^{N-i-1}\xi^j\left(1+\xi^{2(N-i-j)+1}\right)T_{j+i+1,m},$$

$$i = 1, 2, \ldots, N-1. \tag{12}$$

The inner heat transfer coefficient ξ is a dimensionless quantity given by [10]

$$\xi=\frac{\mu}{\mu+2}, \tag{13}$$

where

$$\mu=\frac{\alpha\Delta t}{(\Delta l)^2}, \tag{14}$$

is a dimensionless quantity called the Fourier number for one slab, or simply the mesh ratio, and α is the thermal diffusivity of the slab material. As can be seen from Eq. (12), the new temperature of a particular slab at the time t_{m+1} depends not only on temperatures of its neighboring slabs at time t_m, but also on the temperatures of all slabs in the medium. The influence of more distant slabs is diminishing exponentially.

3. COMPARISON

For $N > 2$ bodies arranged in a straight line, the Fourier's algorithm principally differs from the DHW algorithm. In the DHW algorithm, the wave of redistribution is marching through the medium, consecutively changing the temperatures of neighboring slices, two at the time, while in the Fourier's algorithm, $(N-1)$ thin layers ω move to and fro between the successive bodies, all at once. Nevertheless, from the heat transfer point of view, the two algorithms are identical for $N=2$. From Eq. (5), it follows that the amount of heat transferred by an infinitely small layer δ between the two bodies in Δt is $\Delta l A\rho c\Delta\theta = -(\theta-\vartheta)\delta A\rho c$. From the comparison with the Fourier law,

$$-(\theta-\vartheta)\delta A\rho c = -\lambda\frac{(\theta-\vartheta)}{\Delta l}A\Delta t, \tag{15}$$

it follows that

$$\frac{\delta}{\Delta l} = \frac{\lambda\Delta t}{\rho c\Delta l^2} = \frac{\alpha\Delta t}{\Delta l^2} = \mu. \tag{16}$$

If the terms $(\delta/\Delta l)$ in Eq. (9) are replaced with the mesh ratio μ, the temperatures of N bodies arranged in a straight line become finally

$$\alpha+(\beta-\alpha)\mu,\ \beta+(\alpha-2\beta+\gamma)\mu,\ \ldots,\ \psi+(\chi-\psi)\mu, \tag{17}$$

or, in the notation similar to the DHW algorithm,

$$\begin{aligned} T_{0,m+1} &= T_{0,m}+\mu(T_{1,m}-T_{0,m}),\\ T_{i,m+1} &= T_{i,m}+\mu(T_{i-1,m}-2T_{i,m}+T_{i+1,m}), \quad i=1,2,3,\ldots,N-2,\\ T_{N-1,m+1} &= T_{N-1,m}+\mu(T_{N-2,m}-T_{N-1,m}). \end{aligned} \tag{18}$$

We consider a simple model problem for the heat flow in a finite homogeneous unchanging medium of thickness L, with no heat source. The medium is adiabatically insulated and the initial temperature distribution is given by the Dirac delta function $\delta(x,t)$. The problem for $x\in[0,L]$

and $t \geqslant 0$ is

$$\begin{aligned} u_t &= \alpha u_{xx}, \quad t \geqslant 0, \;\; 0 \leqslant x \leqslant L, \\ u_x(0,t) &= u_x(L,t) = 0, \quad t \geqslant 0, \\ u(x,0) &= \delta(x,t), \quad 0 \leqslant x \leqslant L. \end{aligned} \tag{19}$$

The analytical solution of the problem, Eq. (19), is given by

$$\begin{aligned} u(X, Fo) = {} & \frac{1}{\sqrt{\pi Fo}} \sum_{n=0}^{\infty} \left(\exp\left[\frac{-(2n+X)^2}{4Fo} \right] \right. \\ & \left. + \exp\left[\frac{-(2n+2-X)^2}{4Fo} \right] \right), \end{aligned} \tag{20}$$

where X and Fo are the dimensionless x-coordinate and the Fourier number, respectively, defined as

$$X = \frac{x}{L}, \quad Fo = \frac{\alpha t}{L^2}. \tag{21}$$

To approximate the model, Eq. (19), by both the DHW and Fourier algorithms, the medium will be divided into N equal slabs of thickness $\Delta l = L/N$, with nodal points in the middle of the slabs. The dimensionless nodal point positions,

$$X_i = \frac{x_i}{L}, \quad i = 0, 1, 2, \ldots, N-1, \tag{22}$$

and the dimensionless time,

$$Fo_m = m \Delta Fo = m \frac{\alpha \Delta t}{L^2}, \quad m = 0, 1, 2, \ldots \quad , \tag{23}$$

will be used in graphs. The initial temperature of the first slab will be numerically equal to N, while the rest of the slabs will be at zero temperature at $Fo = 0$.

The temperature distributions for five different Fo calculated using the DHW algorithm for $N = 20$, $\mu = 0.2$, and $\Delta Fo = 5.0 \times 10^{-4}$ are shown in Fig. 2 along with the exact solution in Eq. (20). Errors, $E_{i,m}$, defined as the difference between the approximative and exact temperatures $E_{i,m} = T_{i,m} - u(X_i, Fo_m)$, for $N = 20$, $\mu = 0.2$, and $\Delta Fo = 5.0 \times 10^{-4}$ at $Fo =$ 0.025, 0.05, 0.075, 0.15, 2.0, are shown in Fig. 3. Both algorithms clearly give quite accurate results. The DHW algorithm is more precise than the Fourier algorithm, especially for slabs close to the front of the medium.

If the same calculations are carried out with a refined time step, then the Fourier algorithm results are closer to those of the DHW. The errors for $N = 20$, $\mu = 0.01$, and $\Delta Fo = 2.5 \times 10^{-5}$ are shown in Fig. 4. The time step refinement has no significant effect on the DHW algorithm.

On the contrary, an increase in the number of divisions N has a profound effect on the precision of both algorithms. The error distributions for both algorithms are shown in Fig. 5 for $N = 40$, $\mu = 0.05$, and $\Delta Fo = 3.125 \times 10^{-5}$ at $Fo = 0.025$, 0.05, 0.075, 0.15, 2.0. The errors are now about four times less than those from Fig. 4.

In modern numerical analysis, the system of Eq. (18) represents an explicit finite difference (EFD) scheme, although a pattern of grid points used in today EFD schemes differs from the one used in Fourier's algorithm. It is a well-known fact [16] that the EFD scheme is not stable for the mesh ratio $\mu \geqslant \frac{1}{2}$. From Fig. 6, where two sets of temperature results calculated with the DHW and Fourier algorithms are shown (both for $N = 20$, $\mu = 0.5$, and $\Delta Fo = 1.25 \times 10^{-3}$ at $Fo = 0.025$, 0.05, 0.075, 0.15, 2.0), it is clearly visible that Fourier's algorithm is becoming unstable and the errors E are oscillating. The DHW algorithm's E values for $N = 20$, $\mu = 0.5$, and $\Delta Fo = 1.25 \times 10^{-3}$, shown in Fig. 7, are on the contrary still quite small.

When the DHW imitates diffusion, the upper limit for the inner transfer coefficient is $\xi < 0.5$. It follows from Eqs. (13) and (14) that the upper limit for the mesh ratio in DHW is $\mu \leqslant 2$. This 'stability' criterion is not as strong as in the case of the EFD scheme, because the time step in the DHW is actually subdivided (discretized) to $2N$ sub-steps and oscillations are effectively dumped. This is illustrated in Figs. 8 and 9, where the results of approximative temperature calculations using the DHW algorithm and the error distribution, respectively, are shown for $N = 20$, $\mu = 2.5$, and $\Delta Fo = 6.25 \times 10^{-3}$. With the exception of the first curve for $Fo = 0.025$, all error values are less than 0.005.

4. CONCLUSION

From the comparison of the DHW algorithm and the Fourier algorithm for transmission of heat between discrete bodies we have found:

- For the same number of divisions N and the same mesh ratio μ, the DHW algorithm is generally more precise than the Fourier algorithm.
- For the same number of divisions N, the Fourier algorithm results converge to the DHW algorithm results for the mesh ratio $\mu \to 0$.
- The Fourier algorithm represents the EFD scheme with nodal points in the middle of equal thickness slabs.

- The Fourier algorithm is stable for the mesh ratio $\mu < \frac{1}{2}$, while the DHW algorithm is stable for $\mu < 2$. Even for $\mu > 2$, the DHW algorithm results are not showing signs of oscillations.
- Both algorithms converge to the exact solution for $N \to \infty$ and $\mu \to 0$.

REFERENCES

1. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955).
2. J. Herivel, *Joseph Fourier The Man and The Physicist* (Claredon Press, Oxford, 1975), p. 149.
3. I. Grattan-Guinness and J. R. Ravetz, *Joseph Fourier 1768–1830* (MIT Press, Cambridge, 1972), p. 38.
4. J. B. Biot, *Bibliothèque Britannique*. **37**:310 (1804).
5. J. Fourier, *Théorie de la Propagation de la Chaleur,* printed with comments in I. Grattan-Guinness, *Joseph Fourier 1768–1830* (MIT Press, Cambridge, Massachusetts, 1972)
6. J. Fourier, *Théorie Analytique de la Chaleur* (Firmin Didot, Paris, 1822).
7. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), Chap. IV, Sect. 2
8. J. Herivel, *Joseph Fourier The Man and The Physicist* (Claredon Press, Oxford, 1975), p. 149.
9. J. Gembarovic, M. Löffler, and J. Gembarovic, Jr., *Appl. Math. Model.* **28**:173 (2004).
10. M. Löffler, J. Gembarovic, and J. Gembarovic, Jr., *A New Way of Modelling Transport Processes, in Thermal Conductivity 26 Thermal Expansion 14,* R. B. Dinwiddie and R. Mannello, eds. (DEStech Pubs., Lancaster, Pennsylvania, 2005), pp. 123–133.
11. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), p. 225.
12. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), p. 227.
13. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), p. 228.
14. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), p. 238.
15. J. Fourier, *Analytical Theory of Heat, Translated with notes of A. Freeman* (Dover Pubs., New York, 1955), p. 261.
16. K. W. Morton and F. D. Mayers, *Numerical Solution of Partial Differential Equations* (Cambridge University Press, Cambridge, 2005), p. 14.